

Section 6.5 Exponential Growth and decay

Some natural quantities grow (or decay) at a rate proportional to their size. For example, if $m(t)$ is the mass of a radioactive material at time t , then $m'(t) = k \cdot m(t)$: The material decays at a rate proportional to its mass.

In general, we have $\frac{dy}{dt} = k \cdot y(t)$, where k is a constant.

* if $k > 0$, we have natural growth; if $k < 0$, we have natural decay

* The point is to solve the ODE $\frac{dy}{dt} = ky$ for $y(t)$:

$\frac{dy}{dt} = ky \Rightarrow \frac{dy}{y} = k dt$: this is a separable Equation.

$$\int \frac{dy}{y} = \int k dt \Rightarrow \ln|y| = kt + C \Rightarrow |y| = e^{kt+C}$$

Since y is a quantity, it is always positive. So $|y| = y$.

Thus, $y = e^{kt+C}$. How do we solve for C ?

at $t=0$, i.e. initial time, $y(0) = y_0$, initial quantity.

So, $y_0 = y(0) = e^{k \cdot 0 + C} = e^C$. Thus $e^C = y_0$. This yields

$$y(t) = e^{kt+C} = e^{kt} \cdot e^C = y_0 \cdot e^{kt}. \quad \text{Therefore } \boxed{y(t) = y_0 e^{kt}}$$

. We call " k " the constant relative growth (or decay) rate.

Example: In 1990, US pop = 106.5 mil; In 2012, US pop = 314 mil.

Assume growth rate is proportional to population size. What is the relative growth rate?

Let $t=0$ correspond to 1920; so $t=92$ corresponds to 2012.

$$\frac{dP}{dt} = kP \Rightarrow P(t) = P(0)e^{kt} = 106.5 e^{kt}$$

at $t=92$, $P(92) = 314 = 106.5 e^{92k}$. Solve for k .

$$k = \frac{1}{92} \ln\left(\frac{314}{106.5}\right) \approx 0.0118. \text{ Relative growth of about } 1.18\% \text{ per year}$$

The Final formula for $P(t)$ is $P(t) = 106.5 e^{0.0118t}$.

Radioactive decay: Let $m(t) = m(0)e^{kt}$, $k < 0$ be the mass of a radioactive substance at time t (decay)

Half-life: is the time it takes for a sample of a certain substance to lose half its value. That is, if

$$t_0 \text{ is the Half life, Then } m(t_0) = \frac{1}{2} m(0).$$

Example: Chernobyl residents who were relocated after the blast in 1986 had exposure levels around 350 mSv (milliSievert).

The natural we're all exposed to (per year) is about 2 mSv.

If the radioactive materials of Chernobyl have a Half-life of 7 years, when will the area be habitable again?

Let $t=0$ correspond to 1986. Then $R(t) = 350e^{kt}$, where $R(t)$ is the Radiation level at time t . Half-life = 7 years \Rightarrow

$$R(7) = \frac{1}{2} 350 = 175 = 350e^{7k} \text{ solve for } k. \quad k = \frac{-\ln 2}{7} \approx -0.099$$

Thus $R(t) = 350e^{-0.099t}$. when will $R(t) = 2$?

$$\text{Set } 2 = 350e^{-0.099t}, \text{ and solve for } t \Rightarrow t \approx 52 \text{ years (since 1986)}$$

Newton's Law of cooling: Rate of cooling of an object is proportional to the temperature difference between the object and its

Surroundings: $\frac{dT}{dt} = k(T - T_s)$, $k = \text{constant}$,
 $T_s = \text{surrounding Temp. (constant)}$

Example: A bottle of Soda at room temperature (72°F) is placed in a refrigerator (44°F); a half-hour later, the bottle has cooled to 61°F . Find a formula for the temperature $T(t)$.

$$\frac{dT}{dt} = k(T - T_s) \Rightarrow \int \frac{dT}{T - T_s} = \int k dt \Rightarrow \ln |T - T_s| = kt + C$$

$$\text{Thus, } T - T_s = e^{kt+C} \Rightarrow T(t) = e^C \cdot e^{kt} + T_s.$$

$$\text{At } t=0, \quad T(0) = T_0 = e^C + T_s \Rightarrow e^C = T(0) - T_s$$

$$\text{Thus, } \boxed{T(t) = (T(0) - T_s) e^{kt} + T_s} \quad \text{Here, } T(0) = 72, T_s = 44$$

$$\text{So } T(t) = 28e^{kt} + 44. \quad T(0.5) = 61 = 28e^{0.5k} + 44$$

$$\text{Solve for } k: \quad k = 2 \ln\left(\frac{17}{28}\right) \Rightarrow T(t) = 28e^{2 \ln(17/28)t} + 44.$$

Continuously Compounded Interest.

• If \$1000 is invested in an account paying 5% interest, compounded annually, Then

$$\cdot \text{After 1 year, Value} = 1000(1+0.05) = 1000(1.05) = \$1050$$

$$\cdot \text{After 2 years, Value} = 1050(1.05) = 1000(1.05)^2 = \$1102.5$$

$$\cdot \text{After } t \text{ years, Value} = 1000(1.05)^t = 1000(1+r)^t$$

• If interest is compounded n times a year, for t years, Then
Value = $A_0 \left(1 + \frac{r}{n}\right)^{n \cdot t}$, where A_0 = initial investment.

Question: what if interest is compounded continuously ($n \rightarrow \infty$)?

$$A(t) = \lim_{n \rightarrow \infty} A_0 \left(1 + \frac{r}{n}\right)^{n \cdot t}$$
$$= A_0 \lim_{n \rightarrow \infty} \left[\left(1 + \frac{r}{n}\right)^{\frac{n}{r}} \right]^{r \cdot t}$$

$$= A_0 \left[\lim_{n \rightarrow \infty} \left(1 + \frac{r}{n}\right)^{\frac{n}{r}} \right]^{r \cdot t} \quad \text{let } m = \frac{r}{n}$$

$$= A_0 \left[\lim_{m \rightarrow 0} \left(1 + m\right)^{\frac{1}{m}} \right]^{r \cdot t}$$

$$= A_0 e^{r \cdot t}$$

(Recall, $e = \lim_{m \rightarrow 0} \left(1 + m\right)^{\frac{1}{m}}$)

$$\text{So, } A(t) = A_0 e^{r \cdot t}.$$